General Dispersive Discovery and The Laplace Transform

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This is a guest post by WebHubbleTelescope.

I find it interesting that much of the mathematics of depletion modeling arises from considerations of basic time-series analysis coupled with useful transforms from signal processing. As a case in point, Khebab has postulated how the idea of loglet theory fits into multi-peak production profiles, which have a close relationship to the practical wavelet theory of signal processing. Similarly, the Oil Shock Model uses the convolution of simple data flow transfer functions that we can also express as cascading infinite impulse response filters acting on a stimulated discovery profile. This enables one to use basic time series techniques to potentially extrapolate future oil production levels, in particular using reserve growth models ala Khebab's HSM or the maturation phase DD. [1]

In keeping with this tradition, it turns out that the generalized Dispersive Discovery model fits into a classic canonical mathematical form that makes it very accessible to all sorts of additional time-series and spatial analysis. Actually the transform has existed for a very long while -- just ask the guy to the right.

Much of the basis of this formulation came from a comment discussion originally started by Vitalis and an observation by Khebab (scroll down if curious). I mentioned in the comments that the canonical end result turns into the Laplace transform $g(s) = \int_0^\infty f(t)\, e^{-s\,t}\, dt$ of the underling container volume density; this becomes the aforementioned classic form familiar to many an engineer and scientist. The various densities include an exponential damping (e.g. more finds near the surface), a point value (corresponding to a seam at a finite depth), a uniform density abruptly ending at a fixed depth, and combinations of the above.

The following derivation goes through the steps in casting the dispersive discovery equations into a Laplace transform. The *s* variable in Laplace parlance takes the form of the reciprocal of the dispersed depth, 1/lambda.

> What does the term *lambda* really signify? A fairly good analogy, although not perfect, comes from the dynamics of an endurance race consisting of thousands of competitors of hugely varying skill or with different handicaps. If one considers that at the start of the race, the basic extent of the mob has a fairly narrow spread, roughly equal to the distance traveled. The value of lambda over distance traveled describes this dispersion. I postulate that the dispersion of the mob increases with the average distance that the center of gravity of the mob has traveled. Overall, we empirically observe enough stragglers that the standard deviation of the dispersive spread may to first-order match this average distance. The analogy comes about when we equate the endurance racers to a large group of oil prospectors seeking oil discoveries in different regions of the world. The dispersion term lambda signifies that the same spread in skills (or conversely the difficulty in prospecting equating with certain competitors having to run through mud or while wearing cement boots) would occur in the discovery cycle just like it does in an endurance race. The more varied the difficulties that we as competitors get faced with, the greater the dispersion will become and a significant number of stragglers will always remain. The notion of stragglers then directly corresponds to the downside of a discovery profile -- we will always have discovery stragglers exploring the nooks and crannies of inaccessible parts of the world for oil.

The basic idea behind dispersive discovery assumes that we search through the probability space of container densities, and accumulate discoveries proportional to the total size searched (see the equation derivation in **Figure 1**) . The search depths themselves get dispersed so that values exceeding the cross-section of the container density random variable *x* with the largest of the search variables *h* getting weighted as a potential find. In terms of the math, this shows up as a conditional probability in the 3rd equation, and due to the simplification of the inner integral, it turns into a Laplace transform as shown in the 4th equation.

General Dispersive Discovery - container L, dispersed depth λ

$$p(h, \lambda) = \frac{1}{\lambda} \cdot e^{-h/\lambda}$$

$$P(x|\lambda) = \int_{(h=x)}^{\infty} p(h, \lambda) dh = \int_x^{\infty} \frac{1}{\lambda} \cdot e^{-h/\lambda} dh$$

$$\bar{U}(\lambda, L) = \int_0^{\infty} f(x, L) \cdot P(x|\lambda) dx = \int_0^{\infty} f(x, L) \cdot \left(\int_x^{\infty} \frac{1}{\lambda} \cdot e^{-h/\lambda} dh \right) dx$$

$$\bar{U}(\lambda, L) = \int_0^{\infty} f(x, L) \cdot e^{-x/\lambda} dx$$

$$\left[\begin{array}{ll} \text{if exponential container} & f(x, L_0) = \frac{1}{L_0} \cdot e^{-x/L_0} \\ & \bar{U}(\lambda, L_0) = \frac{1}{1 + \frac{L_0}{\lambda}} \end{array} \right]$$

$$\left[\begin{array}{ll} \text{if point container} & f(x, L_0) = \delta(x - L_0) \\ & \bar{U}(\lambda, L) = e^{-L_0/\lambda} \end{array} \right]$$

$$\left[\begin{array}{ll} \text{if uniform container} & f(x, L_0) = (u(x) - u(x - L_0))/L_0 \\ & \bar{U}(\lambda, L_0) = \lambda \cdot (1 - e^{-L_0/\lambda})/L_0 \end{array} \right]$$

Figure 1: *Fundamental equations describing generalized Dispersive Discovery*

The fun starts when we realize that the container function *f(x)* becomes the target of the Laplace transform. Hence, for any *f(x)* that we can dream up, we can short-circuit much of the additional heavy-duty math derivation by checking first to see if we can find an entry in any of the commonly available Laplace transform tables.

In the square bracketed terms shown after the derivation, I provided a few selected transforms giving a range of shapes for the cumulative discovery function, U-bar. Remember that we still need to substitute the lambda term with a realistic time dependent form. In the case of substituting an exponential growth term for an exponentially distributed container, lambda ~ exp(kt), the first example turns directly into the legendary Logistic sigmoid function that we derived and demonstrated previously.

The second example provides some needed intuition how this all works out. A point container describes something akin to a seam of oil found at a finite depth L_0 below the surface.[2] Note that it takes much longer for the dispersive search to probabilistically "reach" this quantity of oil as illustrated in the following figure. Only an infinitesimal fraction of the fast dispersive searches will reach this point initially as it takes a while for the bulk of the searches to approach the average depth of the seam. I find it fascinating how the math reveals the probability aspects so clearly while we need much hand-waving and subjective reasoning to convince a lay-person that this type of behavior could actually occur.

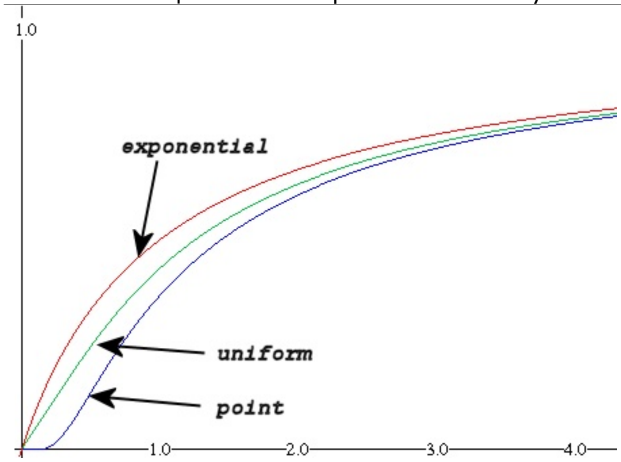


Figure 2: *Cumulative discoveries for different container density distributions analytically calculated from their corresponding Laplace transforms. The curves as plotted assume a constant search rate. An accelerating search rate will make each of the curves more closely resemble the classic S-shaped cumulative growth curve. For an exponentially increasing average search rate, the curve in red (labeled exponential) will actually transform directly into the Logistic Sigmoid curve -- in other words, the classic Hubbert curve.*

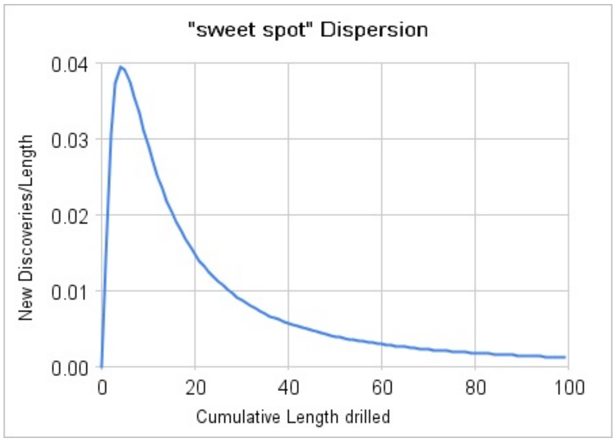
The 3rd example describes the original motivator for the Dispersive Discovery model, that of a rectangular or uniform density. I used the classical engineering unit-step impulse function *u(x)* to describe the rectangular density. As a sanity check, the lookup in the Laplace transform table matches exactly what I derived previously in a non-generalized form, i.e. without the benefit of the transform.

Khebab also suggests that an oil window "sweet spot" likely exists in the real world, which would correspond to a container density function somewhere in between the "seam" container and the other two examples. I suggest two alternatives that would work (and would conveniently provide straightforward analytical Laplace transforms). The first would involve a more narrow uniform distribution that would look similar to the 3rd transform. The second would use a higher order exponential, such as a gamma density that would appear similar to the 1st transform example (see table entry 2d in the Wikipedia Laplace transform table):

$$\frac{1}{(s+\alpha)^{n+1}}$$

Interestingly, this function, under an exponentially increasing search rate will look like a Logistic sigmoid cumulative raised to the n^{th} power, where n is the order of the gamma density! (Have any oil depletion analysts have ever empirically observed such a shape?)

The following figures represent some substantiation for the "sweet spot" theory as it plots Hubbert's original discovery versus cumulative footage chart against one possible distribution -- essentially the Laplace Transform of a Gamma of order-2.



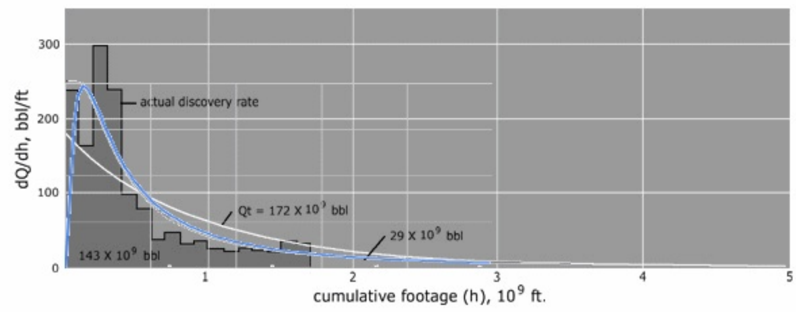


Figure 3: *Derivative of the oil window "sweet spot" Laplace*

transform.

Figure 4: Eyeball fit to Hubbert's cumulative footage data.

The following scatter plots (**Figures 5 to 8**) demonstrate how we can visualize the potential discovery densities. Each one of the densities gets represented by a Monte Carlo simulation of randomized discovery locations. Each dot represents a

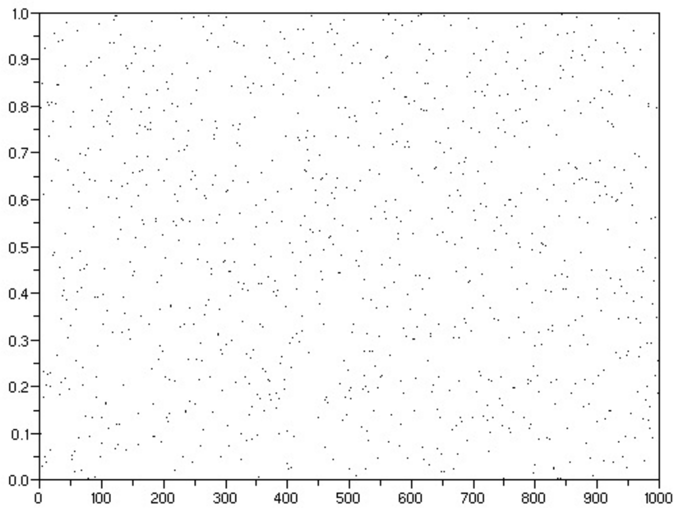


Figure 5: *A uniform density of potential discoveries over a finite volume gives a normalized average value of 0.5. This distribution was the impetus for the[original Dispersive Discovery model](#).*

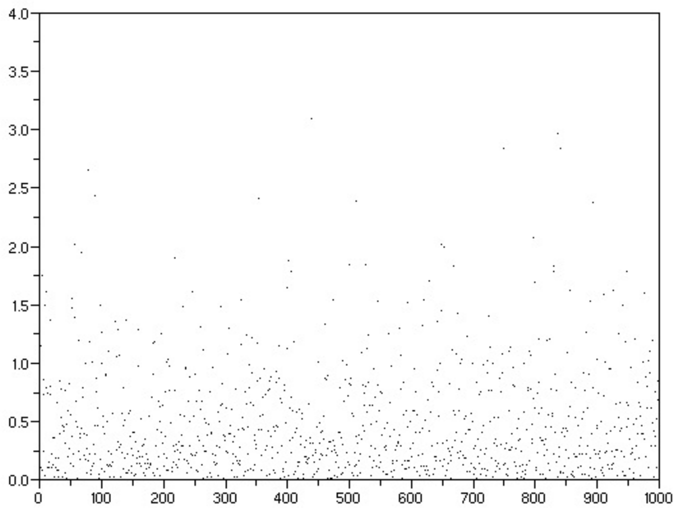


Figure 6: *A damped exponential density of potential discoveries over a finite volume gives a normalized average value of 0.5. When combined with an exponentially accelerating dispersive search rate, this will result in the [Logistic Sigmoid curve](#).*

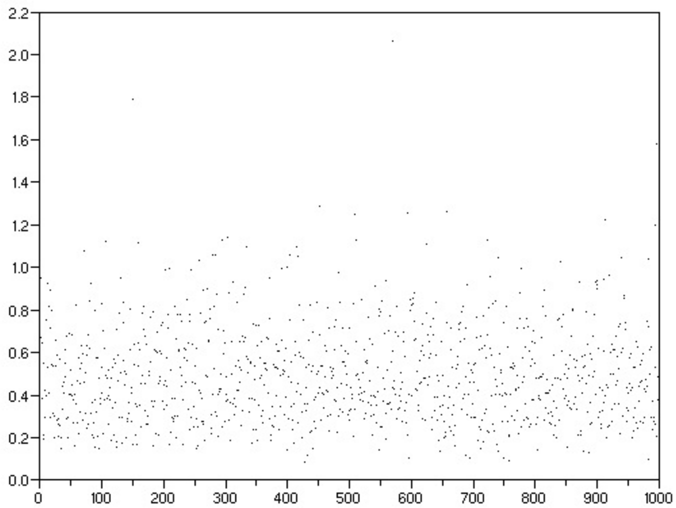


Figure 7: *A gamma order-5 density of potential discoveries over a finite volume narrows the spread around 0.5*

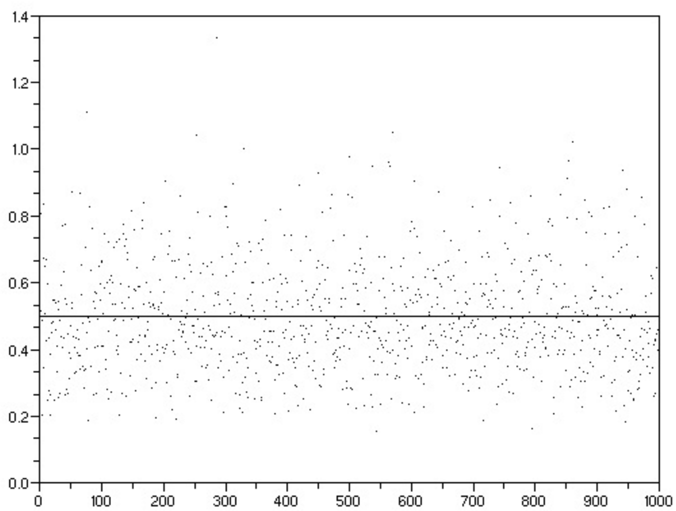


Figure 8: *A gamma order-10 density of potential discoveries over a finite volume further narrows the spread around 0.5. At the limit of even high orders, the density approaches that of the "seam" shown as the solid line drawn at 0.5.*

I discovered an interesting side result independent of the use of any of the distributions. It turns out that the tails of the instantaneous discovery rates (i.e. the first derivative of the cumulative discovery) essentially converge to the same asymptote as shown in **Figure 9**. This has to do more with the much stronger dispersion effect than that of the particular container density function.

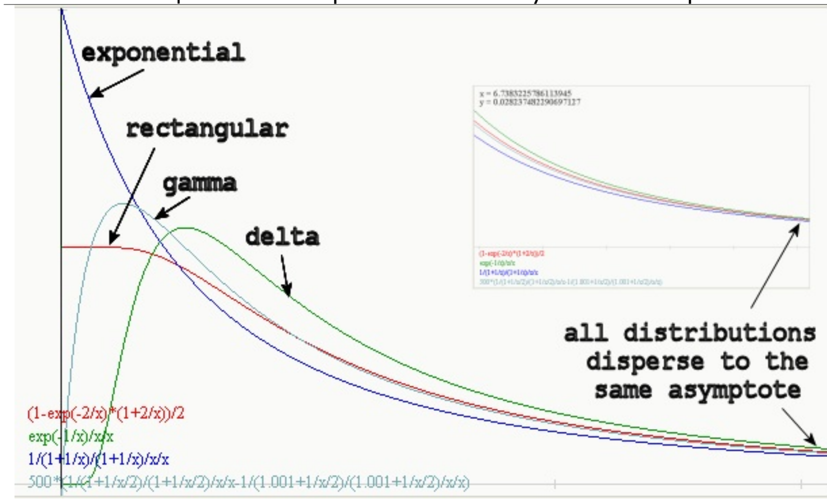


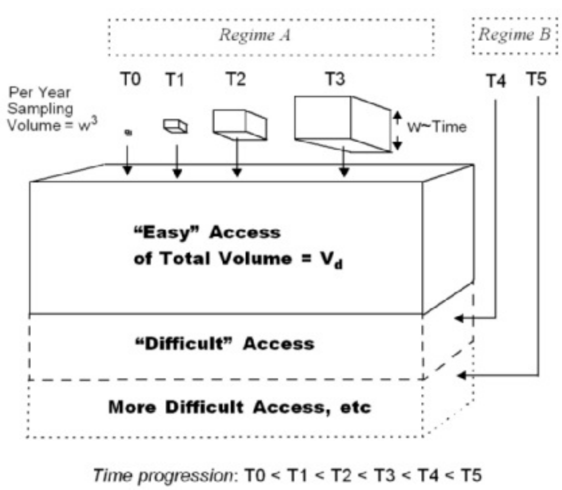
Figure 9: *The set of first derivatives of the Laplace Transforms for various container density functions. Note that for larger dispersed depths (or volumes) that the tails tend to converge to a common asymptote. This implies that the backsides of the peak will generally look alike for a given accelerating search function.*

In summary, using the Laplace Transform technique for analyzing the Dispersive Discovery model works in much the same way as it does in other engineering fields. It essentially provides a widely used toolbox that simplifies much of the heavy-lifting analytical work. It also provides some insight to those analysts that can think in terms of the forbidding and mind-altering reciprocal space. Indeed, if one ponders why this particular model has take this long to emerge (recall that it does derive the Hubbert Logistic model from first principles and it also explains the enigma of reserve growth exceedingly well), you can almost infer that it probably has to do with the left-field mathematical foundation it stems from. After all, I don't think that even the legendary King Hubbert contemplated that a Laplace Transform could describe peak oil ...

Footnotes

[1] I recently posted here how the Oil Shock Model gets represented as a statistical set of *"shocklets"* to aid in unifying with the loglet and HSM and DD approaches

[2] I use depth and volume interchangeably for describing the spatial density. Instead of using depth with a one-dimensional search space, essentially the same result applies if we consider a container volume with the search space emanating in 3 dimensions (see figure to the right). The extra 2 dimensions essentially reinforce the dispersion effects, so that the qualitative and quantitative results remain the same with the appropriate scaling effects. I fall back on the traditional "group theory" argument at this stage to avoid unnecessarily complicating the derivation.



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